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Generalized n-level Jaynes–Cummings and Dicke models, classical rational r-matrices and algebraic Bethe ansatz

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Abstract
We construct integrable n-level generalizations of Jaynes–Cummings and Dicke models corresponding to the simple (reductive) Lie algebras $\mathfrak{g}$ of rank n. We show that for each such Lie algebra there exist many integrable Jaynes–Cummings and Dicke-type models each of which is associated with the reductive subalgebras $\mathfrak{g}_0 \subset \mathfrak{g}$ containing Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and is obtained via reduction from Jaynes–Cummings or Dicke-type models with a maximal number of bosons. We diagonalize the constructed Jaynes–Cummings and Dicke-type Hamiltonians in the physically most interesting case of $\mathfrak{g} = gl(n)$ using a nested algebraic Bethe ansatz.

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1. Introduction

An important quantum mechanical problem is the interaction of the charged matter with radiation (see books [1, 2] for reviews).

The Jaynes–Cummings model [3] is the simplest, but yet the non-trivial model describing the interaction of a two-level atom with an electromagnetic field at the dipole and rotating-wave approximations. Its generalization, the so-called Dicke model [4], describes the same interaction of N two-level atoms with an electromagnetic field. Mathematically, the Jaynes–Cummings model is a system of one boson interacting with a spin, and the Dicke model is a system of one boson interacting with N spins. The most important property of these two models is their *complete quantum integrability*. This feature permits one to find exactly the spectrum and eigenvectors of these models [5, 6] (see also [7]).

The natural problem, interesting both physically and mathematically, is to find integrable models that will (approximately) describe the interaction of the Nn-level atoms with many modes of an electromagnetic field and generalize Dicke and Jaynes–Cummings models. There have been a number of papers [8–13] formulating the corresponding models from the physical considerations in the so-called n-level, $(n-1)$-mode case. Surprisingly, there are only few papers trying to find and analyze possible integrable generalizations of the Dicke and Jaynes–Cummings models [14, 15].

The main idea of the standard approach to the integrability of the Dicke model [5, 6] is to represent it as a special limiting case of the trigonometric Gaudin model [16] describing a system of $N+1$ interacting spins. The substantial role in this limit is played by the so-called Holstein–Primakoff [17] realization of the $so(3) \simeq sl(2)$ spin algebra. This approach to the integrability of Jaynes–Cummings and Dicke models was a true obstacle [15] on a way to obtaining n-level integrable generalization of the Dicke model. Indeed, such a generalization implies utilization the Gaudin model based on $sl(n)$ (or $gl(n)$) Lie algebra. The corresponding limiting procedure should be based on the explicit formulae for the Holstein–Primakoff-type realization of the $sl(n)$ (or $gl(n)$) algebra and the different coadjoint orbits. But such realizations are not known in the general case. Some explicit formulae exist only for the case of the maximally degenerated coadjoint orbits [18, 19]. That is why only the simplest '3-level, 2-mode' generalized integrable Dicke model was analyzed, and its spectrum was found [14].

In the present paper, we show that the above difficulty can be overcome by changing the interpretation of the integrability of Jaynes–Cummings and Dicke models. We propose to consider the Jaynes–Cummings and Dicke models starting not from the trigonometric Gaudin model associated with classical trigonometrical r-matrices [20], but from the rational r-matrix of Yang and the specially chosen quantum Lax operators that satisfy linear r-matrix algebra with this rational r-matrix. This interpretation is quite natural because after the limiting procedure applied to the Lax operators of the trigonometric Gaudin model one obtains a rational Lax operator [6], and it is natural to consider such Lax operators without any limits, starting at once from the rational r-matrix of Yang. Such interpretation permits us to consider the generalized Jaynes–Cummings and Dicke models for the arbitrary semisimple (reductive) Lie algebra $\mathfrak{g}$.

In order to describe the rational quantum Lax operators for the generalized Jaynes–Cummings and Dicke models, we briefly describe the structure of the algebra of Lax operators corresponding to a rational $\mathfrak{g}\otimes\mathfrak{g}$-valued classical r-matrix of Yang. The generalized Jaynes–Cummings model corresponds to the case of the special $\mathfrak{g}$-valued Lax operators $\hat{L}(u)$ having first-order poles at the points $u=0$ and $u=\infty$, and the generalized Dicke model corresponds to the case of the special Lax operators $\hat{L}(u)$ having first-order poles at the points $u=\nu_1,\ldots,u=\nu_N$ and $u=\infty$. In more detail, the Lax operators of the Jaynes–Cummings and Dicke models are obtained from the general Lax operators with the above-described pole structure after the reduction over the subalgebra of central elements. It turned out that there are different possibilities of such reduction that are labeled by the reductive subalgebras $\mathfrak{g}_0\subset\mathfrak{g}$, or, equivalently by a closed, symmetric subset Δ_0 of a set of roots Δ of the algebra $\mathfrak{g}$. In the result, we obtain a set of Jaynes–Cummings and Dicke-type quantum Lax operators labeled by a pair of simple (reductive) Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{g}_0$. The corresponding quantum Hamiltonian of the generalized Dicke model is constructed by using the generating function $\hat{C}^2(u)=\frac{1}{2}\operatorname{tr}\hat{L}^2(u)$ and has (up to a constant) the form

$$\hat{H} = \sum_{\alpha\in(\Delta/\Delta_0)_+}\alpha(-K)\hat{b}^+_\alpha\hat{b}^-_\alpha + g\sum_{\alpha\in(\Delta/\Delta_0)_+}\sqrt{\alpha(-K)}\sum_{l=1}^{N}\left(\hat{b}^+_\alpha\hat{S}^{(l)}_{-\alpha}+\hat{b}^-_\alpha\hat{S}^{(l)}_\alpha\right) + \sum_{l=1}^{N}\sum_{i=1}^{\operatorname{rank}g}((g\nu_l+1)k_i+gc_i)\hat{S}^{(l)}_i + g\sum_{l=1}^{N}\sum_{\alpha\in\Delta_0}c_\alpha\hat{S}^{(l)}_\alpha, \quad (1)$$

where $\hat{b}^{\pm}_{\alpha}$ are bosonic creation–annihilation operators, $\hat{S}^{(l)}_{\pm\alpha}$ are the basic elements of the root spaces of the Lie algebra $\mathfrak{g}$ in some irreducible representation π_l, $\hat{S}^{(l)}_i$ are basic elements of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in the same representation, $K = \sum_{i=1}^{\mathrm{rk}\mathfrak{g}} k_i H_i$ is a constant element of $\mathfrak{h}$, and c_i and c_α are arbitrary constants. The generalized Jaynes–Cummings Hamiltonian is obtained as a partial case of the generalized Dicke Hamiltonian (1) corresponding to the case $N = 1$ and $\nu_1 = 0$.

In the physically important $gl(n)$ case Hamiltonian (1) acquires the following form:

$$\hat{H} = \sum_{i,j=1,i<j}^{n} (k_j - k_i)\hat{b}^{+}_{ij}\hat{b}^{-}_{ij} + g\sum_{l=1}^{N}\sum_{i,j=1,i<j}^{n}\sqrt{(k_j - k_i)}\big(\hat{b}^{+}_{ij}\hat{S}^{(l)}_{ji} + \hat{b}^{-}_{ij}\hat{S}^{(l)}_{ij}\big) + \sum_{l=1}^{N}\sum_{i=1}^{n}((g\nu_l + 1)k_i + gc_i)\hat{S}^{(l)}_{ii}, \tag{2}$$

where the operators $\hat{S}^{(l)}_{ij}$ constitute the representation π_l of the Lie algebra $gl(n)$, and Bose creation–annihilation operators $b^{\pm}_{ij}$ together with a unit operator constitute Heisenberg algebra of the dimension $n^2 - n + 1$, and we put for simplicity $c_\alpha = 0, \alpha \in \Delta_0$. In the degenerated cases, when $k_i = k_j$ for some indices i, j we obtain that the number of the Bose fields in the Lax operators, Hamiltonian and all other integrals decreases. This reflects the above-described reduction. The subalgebra $\mathfrak{g}_0$ in this case coincides with the centralizer of the element of the Cartan subalgebra $K = \sum_{i=1}^{n} k_i X_{ii}$. It is necessary to note that in the most degenerated partial case when $k_i = k_j$ for $i, j \in 2, n$ we recover generalization of the Dicke model constructed from the other considerations in [15].

Let us briefly comment on the physical interpretation of the Hamiltonian (2). For this purpose, it will be instructive to interpret the simplified case of the generalized Jaynes–Cummings model ($N = 1, \nu_1 = 0$) and put for simplicity $c_i = 0$, and π_1 is a fundamental representation of $gl(n)$, i.e. $\hat{S}^{(1)}_{ij} \equiv X_{ij}$, where $(X_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. The corresponding generalized Jaynes–Cummings Hamiltonian

$$\hat{H} = \sum_{i,j=1,i<j}^{n} (k_j - k_i)\hat{b}^{+}_{ij}\hat{b}^{-}_{ij} + g\sum_{i,j=1,i<j}^{n}\sqrt{(k_j - k_i)}\big(\hat{b}^{+}_{ij}X_{ji} + \hat{b}^{-}_{ij}X_{ij}\big) + \sum_{i=1}^{n} k_i X_{ii} \tag{3}$$

may be interpreted as an approximation of an exact Hamiltonian of the interaction of an n-level atom with a set of energies $E_j = k_j, j = 1, n$ with an electromagnetic field. The third summand in the Hamiltonian (3) stands for the proper energy of the atom; the first summand stands for the energy of the electromagnetic field, where $w_{ij} = k_j - k_i$ [1] is the frequency of the transition between the levels j and i of the atom, and $\hat{b}^{\pm}_{ij}$ are the creation–annihilation operators of photon with this frequency. At last, the second summand stands for the interaction between the atom and the electromagnetic field at resonance: it corresponds to the transition of the atom from the state with the energy k_j to the state with the energy k_i (or from the state with the energy k_i to the state with the energy k_j) with simultaneous creation (annihilation) of the photon with the frequency w_{ij}.

Let us return to the question of the quantum integrability of the models with the constructed Hamiltonians. In the classical case, the existence of the Lax matrix $L(u)$ with a linear r-matrix Poisson bracket is equivalent to the integrability of the corresponding Hamiltonian system if the Hamiltonian is constructed from the expansions in the spectral parameter u of the generating functions $C^k(u) = \mathrm{tr}\, L^k(u)$ [21]. In the present moment, there is no proof of the quantum integrability of the classically integrable systems with the linear r-matrix bracket. In the

[1] Here and throughout the paper we put for convenience $\hbar = 1$.

general case, the problem of a proof of commutativity of quantum operators that correspond to the higher order in the elements of the Lax matrix classical integrals is very complicated. It was solved (half-explicitly) only for the case of $\mathfrak{g} = gl(n)$ and classical r-matrix of Yang in [22] using the results from the theory of Yangians [23].

Fortunately, in order to find the spectrum of the constructed Hamiltonians (1)–(3) it is enough to prove commutativity of all second-order integrals that is to prove that

$$[\hat{C}^2(u), \hat{C}^2(v)] = 0.$$

We do this for the case of arbitrary simple (reductive) Lie algebras $\mathfrak{g}$ and arbitrary quantum Lax operators possessing the linear r-matrix bracket with the rational r-matrix of Yang. This result generalizes a similar result of [24] obtained for the case of $\mathfrak{g} = gl(n)$. This opens the way to calculate the spectrum of the model using the nested algebraic Bethe ansatz invented in [25] for the quantum group case and repeated in [26] for the Lie algebraic case. Due to the fact that, in [26], the nested Bethe ansatz was considered only in the case of the Gaudin model, i.e. for the very special choice of the Lax operator $\hat{L}(u)$, we diagonalize $\hat{C}^2(u)$, find its spectrum and Bethe equations for the case of an arbitrary Lax operator with linear r-matrix brackets of Yang in a representation possessing 'vacuum' vector and in the physically most interesting case of $\mathfrak{g} = gl(n)$. In such a way we obtain, in particular, the Bethe equations and the spectrum of the generalized Dicke Hamiltonian (2).

The structure of the present paper is as follows: in section 2, we describe the algebra of the quantum Lax operators and prove the quantum commutativity of the generating function of the second-order integrals. In section 3, we obtain Lax operators and Hamiltonians of the generalized Dicke and Jaynes–Cummings models. At last, in section 4, in the case of $\mathfrak{g} = gl(n)$ we diagonalize the generating function of the second-order quantum integrals using the nested algebraic Bethe ansatz.

2. Quantum integrability and rational *r*-matrices

2.1. *Algebra of Lax operators and rational r-matrices*

Let $\mathfrak{g}$ be a simple Lie algebra or reductive Lie algebra $gl(n)$. Let $\{X^\alpha | \alpha = 1, 2, \ldots, \dim \mathfrak{g}\}$ be some basis in $\mathfrak{g}$ with the commutation relations

$$[X^\alpha, X^\beta] = \sum_{\gamma=1}^{\dim \mathfrak{g}} C^{\alpha\beta}_\gamma X^\gamma. \tag{4}$$

Let us consider the classical r-matrix of Yang:

$$r_{12}(u - v) = \frac{\Omega_{12}}{(u - v)} = \frac{\sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} g_{\alpha\beta} X^\alpha \otimes X^\beta}{(u - v)}, \tag{5}$$

where $g_{\alpha\beta}$ are the components of nondegenerate invariant metric on $\mathfrak{g}$ defined as follows: $g_{\alpha\beta} = (X_\alpha, X_\beta)$, $(,)$ is an invariant scalar product on $\mathfrak{g}$, X_α are elements of $\mathfrak{g}$ dual to the elements X^α: $(X_\alpha, X^\beta) = \delta^\beta_\alpha$.

Having a classical r-matrix it is possible to introduce in the space of the $\mathfrak{g}$-valued functions of u with operator coefficients the structure of a Lie algebra:

$$[\hat{L}_1(u), \hat{L}_2(v)] = -[r_{12}(u - v), \hat{L}_1(u) + \hat{L}_2(v)], \tag{6}$$

where $\hat{L}_1(u) = \hat{L}(u) \otimes 1$, $\hat{L}_2(v) = 1 \otimes \hat{L}(v)$ and $\hat{L}(u) = \sum_{\alpha=1}^{\dim \mathfrak{g}} \hat{L}_a(u) X^\alpha$.

In the component form, we have the following expression:

$$[\hat{L}_\alpha(u), \hat{L}_\beta(v)] = -\frac{\sum_{\gamma=1}^{\dim \mathfrak{g}} C^\gamma_{\alpha\beta}(\hat{L}_\gamma(u) - \hat{L}_\gamma(v))}{u-v}, \tag{7}$$

where $C^\gamma_{\alpha\beta}$ are structure constants of $\mathfrak{g}$ in the dual basis $[X_\alpha, X_\beta] = \sum_{\gamma=1}^{\dim \mathfrak{g}} C^\gamma_{\alpha\beta} X_\gamma$.

Remark 1. The explicit form of $\hat{L}(u)$ as a function of u depends on the concrete physical model under consideration. Below we will be mainly interested in the Lax operators $\hat{L}(u)$ that correspond to the generalized Jaynes–Cummings and Dicke models. But first we will remind about the relations the classical r-matrix structure and quantum integrable systems for the case of the arbitrary rational Lax operators.

2.2. Quantum integrals and classical r-matrix

In the case of a classical Hamiltonian system with a r-matrix Poisson bracket one automatically obtains Poisson-commutative generating functions of the classical integrals using invariant functions of the underlying simple Lie algebras [21]. In the quantum case, the situation is more complicated due to the problem of ordering. That is why one has additionally proved the commutativity of the generating functions of quantum integrals. For our purposes it will be sufficient to prove the commutativity of generating functions of the second-order integrals.

The following theorem holds true.

Theorem 2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra (or reductive Lie algebra $gl(n)$). Let $g^{\alpha\beta} \equiv (X^\alpha, X^\beta)$ be the components of the standard invariant nondegenerate metric on $\mathfrak{g}$. Let us define the following operator:*

$$\hat{C}^2(u) = \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} g^{\alpha\beta} \hat{L}_\alpha(u)\hat{L}_\beta(u), \tag{8}$$

where the operators $\hat{L}_\alpha(u)$, $\hat{L}_\beta(u)$ satisfy the linear r-matrix bracket (7). Then,

$$[\hat{C}^2(u), \hat{C}^2(v)] = 0.$$

Remark 2. Note that in the cases of classical matrix Lie algebras $gl(n)$, $so(n)$ and $sp(n)$ one may simply put that

$$\hat{C}^2(u) = \tfrac{1}{2}\,\mathrm{tr}\,\hat{L}^2(u).$$

Proof. Theorem is proved by direct verification. By direct calculation we obtain

$$\begin{aligned}
[\hat{C}^{(2)}(u), L_\beta(v)] &= \left[\sum_{\alpha,\beta=1}^{\dim \mathfrak{g}} g^{\alpha\delta}\hat{L}_\alpha(u)\hat{L}_\delta(u), \hat{L}_\beta(v)\right] \\
&= \sum_{\alpha,\beta,\gamma,\delta=1}^{\dim \mathfrak{g}} \left(g^{\alpha\delta}\hat{L}_\alpha(u)C^\gamma_{\delta\beta}\frac{(\hat{L}_\gamma(u) - \hat{L}_\gamma(v))}{v-u} + g^{\alpha\delta}C^\gamma_{\alpha\beta}\frac{(\hat{L}_\gamma(u) - \hat{L}_\gamma(v))}{v-u}\hat{L}_\delta(u)\right) \\
&= \sum_{\alpha,\beta,\gamma,\delta=1}^{\dim \mathfrak{g}} C^{\gamma\alpha}_\beta \frac{(\hat{L}_\alpha(u)\hat{L}_\gamma(u) - \hat{L}_\alpha(u)\hat{L}_\gamma(v))}{v-u} + C^{\gamma\delta}_\beta \frac{(\hat{L}_\gamma(u)\hat{L}_\delta(u) - \hat{L}_\gamma(v)\hat{L}_\delta(u))}{v-u} \\
&= \sum_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}} C^{\gamma\alpha}_\beta \frac{(\hat{L}_\alpha(u)\hat{L}_\gamma(v) + \hat{L}_\gamma(v)\hat{L}_\alpha(u))}{u-v},
\end{aligned}$$

where we have used the skew symmetry of $C^{\gamma\alpha}_{\beta}$ in the indices γ and α. By analogous direct calculations, using the above-proved equality, we obtain

$$
\begin{aligned}
[\hat{C}^2(u), \hat{C}^2(v)] &= \left[\hat{C}^2(u), \sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} g^{\beta\delta}\hat{L}_\beta(v)\hat{L}_\delta(v)\right]\\
&= \sum_{\alpha,\beta,\delta=1}^{\dim\mathfrak{g}} C^{\gamma\alpha\delta}\left(\frac{\hat{L}_\alpha(u)\hat{L}_\gamma(v)\hat{L}_\delta(v)}{u-vd} + \frac{\hat{L}_\gamma(v)\hat{L}_\alpha(u)\hat{L}_\delta(v)}{u-v}\right)\\
&\quad + \sum_{\alpha,\beta,\gamma=1}^{\dim\mathfrak{g}} C^{\gamma\alpha\beta}\frac{(\hat{L}_\beta(v)\hat{L}_\alpha(u)\hat{L}_\gamma(v) + \hat{L}_\beta(v)\hat{L}_\gamma(v)\hat{L}_\alpha(u))}{u-v}\\
&= \sum_{\alpha,\beta,\gamma=1}^{\dim\mathfrak{g}} C^{\gamma\alpha\beta}\frac{[\hat{L}_\alpha(u), \hat{L}_\gamma(v)\hat{L}_\beta(v)]}{u-v}\\
&= -\sum_{\alpha,\beta,\gamma,\delta=1}^{\dim\mathfrak{g}} C^{\gamma\alpha\beta}\left(\frac{C^{\delta}_{\alpha\gamma}(\hat{L}_\delta(u)-\hat{L}_\delta(v))\hat{L}_\beta(v)}{(u-v)^2} + \frac{C^{\delta}_{\alpha\beta}\hat{L}_\gamma(v)(\hat{L}_\delta(u)-\hat{L}_\delta(v))}{(u-v)^2}\right)\\
&= -\sum_{\beta,\delta=1}^{\dim\mathfrak{g}} \frac{g^{\beta\delta}(\hat{L}_\delta(u)\hat{L}_\beta(v) - \hat{L}_\beta(v)\hat{L}_\delta(u))}{(u-v)^2}\\
&= -\sum_{\beta,\delta,\gamma=1}^{\dim\mathfrak{g}} \frac{g^{\beta\delta}C^{\gamma}_{\beta\delta}(\hat{L}_\gamma(u)-\hat{L}_\gamma(v))}{(u-v)^3} = 0,
\end{aligned}
$$

where $C^{\gamma\alpha\beta} = \sum_{\beta=1}^{\dim\mathfrak{g}} g^{\beta\delta}C^{\gamma\alpha}_{\beta}$, and we have used first a skew symmetry of $C^{\gamma\alpha\beta}$ in all indices, the fact that $\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}} C^{\alpha\beta\gamma}C^{\delta}_{\alpha\gamma} = g^{\beta\delta}$, skew symmetry of $C^{g}_{\beta\delta}$ in the indices β and δ and symmetry of $g^{\beta\delta}$ in these indices.

Theorem is proved. □

In the end of this subsection, let us consider a first-order generating function:

$$\hat{C}^1(u) = \operatorname{tr}\hat{L}(u).$$

By direct calculation one can prove the following proposition.

Proposition 2.1. *Let $\mathfrak{g} = gl(n)$ and the operators $\hat{L}_1(u)$, $\hat{L}_2(u)$ satisfy the linear r-matrix bracket (6) with the classical r-matrix of Yang. Then $\hat{C}^1(u)$ is a generating function of linear Casimir operators of the bracket (6), i.e.*

$$[\hat{C}^1(u), \hat{L}_\alpha(v)] = 0.$$

Remark 3. We will use the property of $\hat{C}^1(u)$ to be a generating function of Casimir operators while diagonalizing the generating function of the quantum integrals $\hat{C}^2(u)$ by means of the nested algebraic Bethe ansatz.

3. Generalized J–C and Dicke models

3.1. The generalized Jaynes–Cummings model

In this subsection, we will introduce generalized Jaynes–Cummings models. For this purpose we will first consider the corresponding Lax operators. In more detail, let a Lax operator have first-order poles at the points $u = 0$ and $u = \infty$:

$$\hat{L}(u) = u\hat{L}^{(-2)} + \hat{L}^{(-1)} + u^{-1}\hat{L}^{(0)}, \qquad \text{where} \quad L^{(k)} = \sum_{\alpha=1}^{\dim \mathfrak{g}} \hat{L}_a^{(k)} X^\alpha. \tag{9}$$

The corresponding commutation relations for the components of the operators $\hat{L}^{(k)}$ are deduced from the commutation relations (7) and have the following explicit form:

$$\left[\hat{L}_\alpha^{(0)}, \hat{L}_\beta^{(0)}\right] = \sum_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}} C_{\alpha\beta}^\gamma \hat{L}_\gamma^{(0)}, \tag{10a}$$

$$\left[\hat{L}_\alpha^{(-1)}, \hat{L}_\beta^{(-1)}\right] = -\sum_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}} C_{\alpha\beta}^\gamma \hat{L}_\gamma^{(-2)}, \tag{10b}$$

$$\left[\hat{L}_\alpha^{(-1)}, \hat{L}_\beta^{(-2)}\right] = \left[\hat{L}_\alpha^{(-2)}, \hat{L}_\beta^{(-2)}\right] = 0, \tag{10c}$$

$$\left[\hat{L}_\alpha^{(0)}, \hat{L}_\beta^{(-2)}\right] = \left[\hat{L}_\alpha^{(0)}, \hat{L}_\beta^{(-1)}\right] = 0. \tag{10d}$$

Bracket (10*a*) means that the quantum operators $\left\{\hat{L}_\alpha^{(0)}\right\}$ constitute a Lie algebra isomorphic to $\mathfrak{g}$. Brackets (10*b*) and (10*c*) show that the quantum operators $\left\{\hat{L}_\alpha^{(-1)}\right\}, \left\{\hat{L}_\alpha^{(-2)}\right\}$ constitute the Lie algebra of a Heisenberg type with the variables $\hat{L}_\alpha^{(-2)}$ being central. The equalities (10*d*) show that the whole algebra of Lax operators is, in this case, the direct sum of the Lie algebra $\mathfrak{g}$ and a Heisenberg-type Lie algebra.

The generating function of the second-order quantum integrals has the form

$$\hat{C}^2(L(u)) = \sum_{k=-2}^{2} u^k \hat{C}_k^2,$$

$$\hat{C}_2^2 = \frac{1}{2}(\hat{L}^{(-2)}, \hat{L}^{(-2)}), \quad \hat{C}_1^2 = (\hat{L}^{(-2)}, \hat{L}^{(-1)}), \quad \hat{C}_0^2 = \frac{1}{2}(\hat{L}^{(-1)}, \hat{L}^{(-1)}) + (\hat{L}^{(-2)}, \hat{L}^{(0)}),$$

$$\hat{C}_{-1}^2 = (\hat{L}^{(0)}, \hat{L}^{(-1)}), \quad \hat{C}_{-2}^2 = \frac{1}{2}(\hat{L}^{(0)}, \hat{L}^{(0)}),$$

where $(\, , \,)$ is an invariant bilinear form on $\mathfrak{g}$, and $C^2(\hat{L}(u)) \equiv \frac{1}{2}(\hat{L}(u), \hat{L}(u))$.

Hamiltonians $\hat{C}_{-2}^2$, $\hat{C}_1^2$ and $\hat{C}_2^2$ are easily shown to be the Casimir operators of the bracket (10*a*)–(10*d*). Operators $\hat{C}_0^2$ and $\hat{C}_{-1}^2$ are the non-trivial quantum integrals. As we will show, they coincide with two second-order integrals of the generalized J–C models.

Note that the Lie brackets (10*b*)–(10*d*) contain many central elements, and we may reduce many of the quantum operators $\hat{L}^{(-2)}$ and $\hat{L}^{(-1)}$ from our system. Due to the fact that the operators $\hat{L}_\alpha^{(-2)}$ constitute a center of our Lax algebra, we may put them to be equal to zero or to be proportional to the unit operator. Now we will show that depending on this reduction the algebra of Lax operators (10) acquires an even larger center.

Proposition 3.1. *Let* $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, *be a triangular decomposition of the Lie algebra* $\mathfrak{g}$, *i.e.* $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 1}, [\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 1}$ *such that* $\mathfrak{g}_0$ *is reductive. Let the value* *of central elements* $\hat{L}_{\alpha}^{(-2)}$ *be chosen in such a way that* $\hat{L}^{(-2)}|_{\mathfrak{g}_{\pm 1}} = 0$, $\hat{L}^{(-2)}|_{[\mathfrak{g}_0,\mathfrak{g}_0]} = 0$. *Then* $\hat{L}^{(-1)}|_{\mathfrak{g}_0}$ *belongs to the center of the Lax algebra (10).*

Proof. Let us first note that the Lax algebra (10) is isomorphic to the quotient algebra $\mathfrak{g}_{0,2} = \widetilde{\mathfrak{g}}_0/J_{0,2}$, where $\widetilde{\mathfrak{g}}_0$ is a loop space $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes Pol(u, u^{-1})$ with the so-called direct-difference brackets $[,]_0$ corresponding to the decomposition: $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$, $\widetilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes Pol(u)$, $\widetilde{\mathfrak{g}}_- = \mathfrak{g} \otimes u^{-1} Pol(u^{-1})$ [21] and $J_{P,Q} = J_P \ominus J_Q$, $J_P = \mathfrak{g} \otimes u^{P+1} Pol(u)$, $J_Q = \mathfrak{g} \otimes u^{-(Q+1)} Pol(u^{-1})$ are ideals in $\widetilde{\mathfrak{g}}_{\pm}$.

Using the introduced triangular decomposition of $\mathfrak{g}$ we obtain in the quotient algebra $\mathfrak{g}_{0,2}$ the following commutation relations:

$$\left[\mathfrak{g}_0^{(-1)}, \mathfrak{g}_0^{(-1)}\right] \subset \mathfrak{g}_0^{(-2)}, \qquad \left[\mathfrak{g}_0^{(-1)}, \mathfrak{g}_{\pm 1}^{(-1)}\right] \subset \mathfrak{g}_{\pm 1}^{(-2)},$$

where $\mathfrak{g}_0^{(-1)} = u^{-1}\mathfrak{g}_0$, $\mathfrak{g}_1^{(-1)} = u^{-1}\mathfrak{g}_1$, $\mathfrak{g}_0^{(-2)} = u^{-2}\mathfrak{g}_0$, $\mathfrak{g}_1^{(-2)} = u^{-2}\mathfrak{g}_1$. From these relations it becomes clear that after factorizing the corresponding Lie algebra over the central elements $\mathfrak{g}_{\pm 1}^{(-2)}$ and elements of $\mathfrak{g}_0^{(-2)}$ of the form $u^{-2}[\mathfrak{g}_0, \mathfrak{g}_0]$ we will have that $\mathfrak{g}_0^{(-1)}$ also becomes the center in this quotient algebra.

This proves the proposition. □

Using this proposition we can prove the following lemma.

Lemma 3.1. *Let the algebra* $\mathfrak{g}$ *be semisimple (reductive). Let* Δ *be the set of its roots,* Δ_+ *be the set of its positive roots,* $\mathfrak{g}_\alpha$ *be the root space,* X_α *be its basis element, and* H_i *be a basis vector of the Cartan subalgebra* $\mathfrak{h}$. *Let* $\Delta_0 \subset \Delta$ *be a closed, symmetric subset of the set of all roots. Let the element* $K = \sum_{i=1}^{\text{rank}\, g} k_i H_i$ *be such that* $\alpha(K) = 0$ *for* $\alpha \in \Delta_0$. *Then the following algebra-valued operators*

$$\hat{L}^{(-2)} = \sum_{i=1}^{\text{rank}\, g} k_i H_i, \qquad \hat{L}^{(-1)} = \sum_{i=1}^{\text{rank}\, g} c_i H_i + \sum_{\alpha\in\Delta_0} c_\alpha X_\alpha + \sum_{\alpha\in(\Delta/\Delta_0)_+} \hat{a}_\alpha^+ X_{-\alpha} + \sum_{\alpha\in(\Delta/\Delta_0)_+} \hat{a}_\alpha^- X_\alpha,$$

$$\hat{L}^{(0)} = \sum_{i=1}^{\text{rank}\, g} \hat{S}_i H_i + \sum_{\alpha\in\Delta_+} \hat{S}_\alpha X_{-\alpha} + \sum_{\alpha\in\Delta_+} \hat{S}_{-\alpha} X_\alpha,$$

where c_i, c_α *and* k_i *are constants,* $\hat{S}_{\pm\alpha}$, $\hat{S}_i$ *are the components of the generalized spin operator in a root basis, and* $\hat{a}_\alpha^\pm$ *are Bose-type creation and annihilation operators:*

$$[\hat{S}_\alpha, \hat{S}_{-\alpha}] = \sum_{i=1}^{\text{rank}\, g} \hat{S}_i \alpha(H_i),$$

$$[\hat{S}_\alpha, \hat{S}_\beta] = N_{\alpha,\beta} \hat{S}_{\alpha+\beta}, \qquad \textit{where} \quad N_{\alpha,\beta} = 0 \quad \textit{if} \quad \alpha+\beta \notin \Delta$$

$$\left[\hat{S}_i, \hat{S}_\alpha^\pm\right] = \alpha(H_i)\hat{S}_\alpha^\pm,$$

$$\left[\hat{a}_\alpha^+, \hat{a}_\beta^-\right] = -\alpha(K)\delta_{\alpha\beta}\mathbf{1},$$

$$\left[\hat{a}_\alpha^+, \hat{a}_\beta^+\right] = [\hat{a}_\alpha^-, \hat{a}_\beta^-] = 0,$$

satisfy the commutation relations of the Lax algebra (10).

Proof. Let $\Delta_0 \subset \Delta$ be a closed, symmetric subset of the set of all roots. Let us define a subspace $\mathfrak{g}_0$ in $\mathfrak{g}$ in the following way: $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\alpha\in\Delta_0} \mathfrak{g}_\alpha$. From the general theory of semisimple Lie algebras it follows that this subspace is a reductive subalgebra of $\mathfrak{g}$, and each reductive subalgebra of $\mathfrak{g}$ containing Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can be obtained in this way. It also follows that the complementary space to $\mathfrak{g}_0$ consists of $\mathfrak{g}_1 = \sum_{\alpha\in(\Delta/\Delta_0)_+} \mathfrak{g}_\alpha$ and $\mathfrak{g}_{-1} = \sum_{\alpha\in(\Delta/\Delta_0)_-} \mathfrak{g}_\alpha$, and $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$ is the triangular decomposition of $\mathfrak{g}$. Moreover, the space $\mathfrak{g}_0/[\mathfrak{g}_0, \mathfrak{g}_0]$ coincides with the center $\mathfrak{z}_0$ of the reductive subalgebra $\mathfrak{g}_0$ and belongs to the Cartan subalgebra $\mathfrak{h}$. The corresponding subspace in $\mathfrak{h}$ consists of the elements that are 'orthogonal' to $\alpha \in \Delta_0$: $\alpha(h) = 0$.

Let us now choose constant central elements (in our Lax algebra) $\hat{L}^{(-2)} = K = \sum_{i=1}^{\mathrm{rank}\, g} k_i H_i$, where constants k_i are not arbitrary but such that $\alpha(K) = 0$ for $\alpha \in \Delta_0$.

Then, as follows from the previous proposition, for such defined central elements $\hat{L}^{(-2)}$, the operator $\hat{L}^{(-1)}|_{\mathfrak{g}_0}$ also belongs to the center of the Lax algebra and, hence, we may put that $\hat{L}^{(-1)}|_{\mathfrak{g}_0} = C = \sum_{i=1}^{\mathrm{rank}\, g} c_i H_i + \sum_{\alpha\in\Delta_0} c_\alpha X_\alpha$, where constants c_i, c_α are arbitrary. Substituting these data into the commutation relations (10) we obtain for the nonconstant elements of the Lax operator the commutation relations described in the lemma. □

Lemma is proven.

Remark 4. The operators $\hat{a}^+_\alpha, \hat{a}^-_\alpha$ constitute a Lie algebra isomorphic to the ordinary Heisenberg algebra. The isomorphism is established by the rescaling of variables: $\hat{a}^+_\alpha \equiv (-\alpha(K))^{1/2}\hat{b}^+_\alpha, \hat{a}^-_\alpha \equiv (-\alpha(K))^{1/2}\hat{b}^-_\alpha$, where the Lie brackets of variables $\hat{b}^\pm_\alpha$ are canonical:

$$\left[\hat{b}^+_\alpha, \hat{b}^-_\beta\right] = \delta_{\alpha\beta}\mathbf{1}, \qquad \left[\hat{b}^+_\alpha, \hat{b}^+_\beta\right] = \left[\hat{b}^-_\alpha, \hat{b}^-_\beta\right] = 0.$$

Example 1. Let $\mathfrak{g} = gl(n)$. In this case, we have the natural basic elements X_{ij}, which are $n \times n$ matrices with the matrix elements $(X_{ij})_{ab} = \delta_{ia}\delta_{jb}$ and commutation relations:

$$[X_{ij}, X_{kl}] = \delta_{kj}X_{il} - \delta_{il}X_{kj}.$$

The Cartan subalgebra coincides with the algebra of diagonal matrices, and $H_i \equiv X_{ii}$, $i \in 1, 2, \ldots, n$ is its orthonormal basis. The set of all roots coincides with the linear forms $\alpha_{ij} = w_i - w_j$, where $w_i(H_j) = \delta_{ij}$, and the corresponding properly normalized element of the root space $\mathfrak{g}_{\alpha_{ij}}$ is $X_{\alpha_{ij}} \equiv X_{ij}$ $(i \neq j)$. An arbitrary element of the Cartan subalgebra has the forms $K = \sum_{i=1}^n k_i X_{ii}$ and $\alpha_{ij}(K) = k_i - k_j$.

The reductive subalgebras containing Cartan subalgebra are $gl(n)_K = gl(n_1) \oplus gl(n_2) \oplus \cdots \oplus gl(n_k)$, where $n_1 + n_2 + \cdots + n_k = n$. They coincide with the centralizers of the element K such that $k_i = k_j$ for $i, j \in m_l + 1, \ldots, m_{l+1}$, where $0 = m_0 < m_1 < m_2 < \cdots < m_k = n, n_k = m_k - m_{k-1}$ and $l \in 0, 1, \ldots, k-1$. In this case we have $\Delta_K = \{\alpha_{ij} | i, j \in m_l + 1, \ldots, m_{l+1}\}$.

A reductive subalgebra of the maximal dimension is $gl(n)_0 = gl(1) \oplus gl(n-1)$. It corresponds to the case $k_2 = k_3 = \cdots = k_n, k_1 \neq k_2$. A reductive subalgebra of the minimal dimension is $gl(n)_0 = gl(1) \oplus gl(1) \oplus \cdots \oplus gl(1)$. It degenerates into an Abelian subalgebra coinciding with a Cartan subalgebra and corresponds to the case $k_i \neq k_j$ for $k_i, k_j \in 1, 2, \ldots, n$.

Let us return to the non-trivial integrals of the Jaynes–Cummings model, $\hat{C}^2_0$ and $\hat{C}^2_{-1}$. They acquire (for that described in the lemma Lax operators) the following form:

$$\hat{C}^2_0 = \sum_{i=1}^{\mathrm{rank}\, g} k_i \hat{S}_i + \sum_{\alpha\in(\Delta/\Delta_0)_+} \frac{1}{2}\left(\hat{a}^+_\alpha \hat{a}^-_\alpha + \hat{a}^-_\alpha \hat{a}^+_\alpha\right) + \frac{1}{2}\sum_{i=1}^{\mathrm{rank}\, g} c_i^2 + \sum_{\alpha\in(\Delta_0)_+} c_\alpha c_{-\alpha}, \quad (11)$$

$$\hat{C}^2_{-1} = \sum_{i=1}^{\mathrm{rank}\, g} c_i \hat{S}_i + \sum_{\alpha\in\Delta_0} c_\alpha \hat{S}_\alpha + \sum_{\alpha\in(\Delta/\Delta_0)_+} \left(\hat{a}^+_\alpha \hat{S}_{-\alpha} + \hat{a}^-_\alpha \hat{S}_\alpha\right), \quad (12)$$

where we have used that $\{H_i\}$ is an orthogonal basis in $\mathfrak{h}$, and a basis element X_α of the space $\mathfrak{g}_\alpha$ is normalized so that $(X_\alpha, X_{-\alpha}) = 1$.

A linear combination of these integrals $\hat{H} = \hat{C}_0^2 + g\hat{C}_{-1}^2$ gives a Hamiltonian of the generalized Jaynes–Cummings model associated with a Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{g}_0$. In terms of the canonical creation–annihilation operators $\hat{b}_\alpha^\pm \equiv (-\alpha(K))^{-1/2}\hat{a}_\alpha^\pm$ it has (up to a constant) the following explicit form:

$$\hat{H} = \sum_{\alpha\in(\Delta/\Delta_0)_+} \left(\alpha(-K)\hat{b}_\alpha^+\hat{b}_\alpha^- + g\sqrt{\alpha(-K)}\left(\hat{b}_\alpha^+\hat{S}_{-\alpha}^- + \hat{b}_\alpha^-\hat{S}_\alpha\right)\right) + \sum_{j=1}^{\text{rank } g}(gc_j + k_j)\hat{S}_j + \sum_{\alpha\in\Delta_0} gc_\alpha\hat{S}_\alpha. \tag{13}$$

This Hamiltonian describes the interaction of the $m = \text{Ord}\,\Delta/\Delta_0$ bosons with the generalized spin operators associated with the Lie algebra $\mathfrak{g}$.

Example 2. Let us consider the case $\mathfrak{g} = gl(n)$ (see example 1) and the case of the generic element of the Cartan subalgebra $K = \sum_{i=1}^n k_i X_{ii}$ such that $k_i \neq k_j$ for $i, j \in 1, 2, \ldots, n$. The corresponding 'generic' Jaynes–Cummings Hamiltonian is

$$\hat{H} = \sum_{i,j=1,i<j}^n (k_j - k_i)\hat{b}_{ij}^+\hat{b}_{ij}^- \pm g \sum_{i,j=1,i<j}^n \sqrt{(k_j - k_i)}\left(\hat{b}_{ij}^+\hat{S}_{ji} + \hat{b}_{ij}^-\hat{S}_{ij}\right) + \sum_{i=1}^n (k_i + gc_i)\hat{S}_{ii}, \tag{14}$$

where the generalized spin operators have Lie brackets isomorphic to $gl(n)$ and $b_{ij}^\pm$ together with the unit constitute Heisenberg algebra of the dimension $n^2 - n + 1$:

$$[\hat{S}_{ij}, \hat{S}_{kl}] = \delta_{kj}\hat{S}_{il} - \delta_{il}\hat{S}_{kj}, \tag{15}$$

$$\left[\hat{b}_{ij}^+, \hat{b}_{kl}^-\right] = \delta_{ik}\delta_{jl}\mathbf{1}, \tag{16}$$

$$\left[\hat{b}_{ij}^+, \hat{b}_{kl}^+\right] = \left[\hat{b}_{ij}^-, \hat{b}_{kl}^-\right] = \left[\hat{b}_{ij}^+, \hat{S}_{kl}\right] = \left[\hat{b}_{ij}^-, \hat{S}_{kl}\right] = 0. \tag{17}$$

Example 3. Let $\mathfrak{g} = gl(n)$ and $gl(n)_0 = gl(1) \oplus gl(n-1)$. The element K centralized by this subalgebra has the form $K = \sum_{i=1}^n k_i X_{ii}$, $k_2 = k_3 = \cdots = k_n$, $k_1 \neq k_2$. The corresponding 'reduced' Jaynes–Cummings Hamiltonian reads as follows:

$$\hat{H} = (k_2 - k_1)\sum_{i=2}^n \hat{b}_{1i}^+\hat{b}_{1i}^- + g\sqrt{(k_2 - k_1)}\sum_{i=2}^n \left(\hat{b}_{1i}^+\hat{S}_{i1} + \hat{b}_{1i}^-\hat{S}_{1i}\right) + (k_1 + gc_1)\hat{S}_{11} + k_2\sum_{i=2}^n \hat{S}_{ii} + g\sum_{i,j=2}^n c_{ij}\hat{S}_{ij}, \tag{18}$$

where $\hat{b}_{1j}^\pm$ and the unit operator form Heisenberg algebra of the dimension $2n - 1$:

$$\left[\hat{b}_{1j}^+, \hat{b}_{1l}^-\right] = \delta_{jl}\mathbf{1},$$
$$[\hat{S}_{ij}, \hat{S}_{kl}] = \delta_{kj}\hat{S}_{il} - \delta_{il}\hat{S}_{kj},$$
$$\left[\hat{b}_{1j}^+, \hat{b}_{1l}^+\right] = \left[\hat{b}_{1j}^-, \hat{b}_{1l}^-\right] = \left[\hat{b}_{1j}^+, \hat{S}_{kl}\right] = \left[\hat{b}_{1j}^-, \hat{S}_{kl}\right] = 0.$$

In the case $c_{ij} = 0$ it corresponds to the so-called n-level, $n - 1$-mode generalized Jaynes–Cummings model [12, 13].

3.2. The generalized Dicke model

Let us now consider the generalized Dicke model. For this purpose we will introduce the corresponding Lax matrix $\hat{L}(u)$. It will be similar to the Lax operator of the generalized Jaynes–Cummings models but will have many poles. In more detail, let us consider Lax operators having first-order poles at the $N+1$ points $u = \infty, \nu_1, \ldots, \nu_N$:

$$\hat{L}(u) = u\hat{L}^{(-2)} + \hat{L}^{(-1)} + \sum_{k=1}^{N} \frac{\hat{L}^{(k)}}{u - \nu_k}, \qquad \text{where} \quad \hat{L}^{(m)} = \sum_{\alpha=1}^{\dim \mathfrak{g}} \hat{L}_a^{(m)} X^{\alpha}. \tag{19}$$

Using commutation relations (7) we obtain the following Lie brackets for the components of the considered quantum Lax operators:

$$\left[\hat{L}_{\alpha}^{(k)}, \hat{L}_{\beta}^{(k)}\right] = \sum_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}} C_{\alpha\beta}^{\gamma} \hat{L}_{\gamma}^{(k)}, \tag{20a}$$

$$\left[\hat{L}_{\alpha}^{(-1)}, \hat{L}_{\beta}^{(-1)}\right] = -\sum_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}} C_{\alpha\beta}^{\gamma} \hat{L}_{\gamma}^{(-2)}, \tag{20b}$$

$$\left[\hat{L}_{\alpha}^{(-1)}, \hat{L}_{\beta}^{(-2)}\right] = \left[\hat{L}_{\alpha}^{(-2)}, \hat{L}_{\beta}^{(-2)}\right] = 0, \tag{20c}$$

$$\left[\hat{L}_{\alpha}^{(k)}, \hat{L}_{\beta}^{(l)}\right] = 0 \qquad \text{if} \quad k \neq l. \tag{20d}$$

Bracket (20*a*) means that the linear operators $\{\hat{L}_{\alpha}^{(k)}\}$ constitute a Lie algebra isomorphic to $\mathfrak{g}$ for each $k \in 1, 2, \ldots, N$. Brackets (20*b*) and (20*c*) mean that the linear operators $\{\hat{L}_{\alpha}^{(-1)}\}, \{\hat{L}_{\alpha}^{(-2)}\}$ constitute a Lie algebra of Heisenberg type with the variables $\hat{L}_{\alpha}^{(-2)}$ being central. The equalities (20*d*) show that the whole algebra of Lax operators is in this case a direct sum of the N copies of the Lie algebras $\mathfrak{g}$ and the Heisenberg-type Lie algebra.

Due to the explicit form of the Lax operator (19), the generating function of the second order has the form

$$C^{(2)}(\hat{L}(u)) = u^2 \hat{C}_2^2 + u\hat{C}_1^2 + \hat{C}_0^2 + \sum_{k=1}^{N} \frac{\hat{C}_{-k}^2}{(u - \nu_k)} + \sum_{k=1}^{N} \frac{\hat{C}_{-2k}^2}{(u - \nu_k)^2},$$

and produces the mutually commuting quantum operators

$$\hat{C}_2^2 = \frac{1}{2}(\hat{L}^{(-2)}, \hat{L}^{(-2)}), \quad C_1^2 = (\hat{L}^{(-2)}, \hat{L}^{(-1)}), \quad C_0^2 = \frac{1}{2}(\hat{L}^{(-1)}, \hat{L}^{(-1)}) + \sum_{k=1}^{N} (\hat{L}^{(-2)}, \hat{L}^{(k)}),$$

$$\hat{C}_{-k}^2 = (\hat{L}^{(-1)}, \hat{L}^{(k)}) + \nu_k (\hat{L}^{(-2)}, \hat{L}^{(k)}) + \sum_{l=1, l\neq k}^{N} \frac{(\hat{L}^{(l)}, \hat{L}^{(k)})}{(\nu_l - \nu_k)}, \quad \hat{C}_{-2k}^2 = \frac{1}{2}(\hat{L}^{(k)}, \hat{L}^{(k)}).$$

Operators $\hat{C}_2^2, \hat{C}_1^2, \hat{C}_{-2k}^2$ are Casimir operators. Operators $\hat{C}_0^2$ and $\hat{C}_{-k}^2$ are non-trivial quantum integrals. As we will show, they coincide with the second-order integrals of generalized Dicke models.

In order to obtain generalized Dicke models, it is enough to repeat the same reduction procedure as in the case of the generalized Jaynes–Cummings model. After such a procedure we will have the following components of the quantum Lax operator $\hat{L}(u)$, defined for the N

copies of semisimple (reductive) Lie algebra $\mathfrak{g}$, element $K = \sum_{i=1}^{\operatorname{rank} g} k_i H_i$ and subset of roots Δ_0 such that $\alpha(K) = 0$ for $\alpha \in \Delta_0$:

$$\hat{L}^{(-2)} = \sum_{i=1}^{\operatorname{rank} g} k_i H_i, \hat{L}^{(-1)} = \sum_{i=1}^{\operatorname{rank} g} c_i H_i + \sum_{\alpha\in\Delta_0} c_\alpha X_\alpha + \sum_{\alpha\in(\Delta/\Delta_0)_+} \hat{a}_\alpha^+ X_{-\alpha} + \sum_{\alpha\in(\Delta/\Delta_0)_+} \hat{a}_\alpha^- X_\alpha,$$

$$\hat{L}^{(k)} = \sum_{i=1}^{\operatorname{rank} g} \hat{S}_i^{(k)} H_i + \sum_{\alpha\in\Delta_+} \hat{S}_\alpha^{(k)} X_{-\alpha} + \sum_{\alpha\in\Delta_+} \hat{S}_{-\alpha}^{(k)} X_\alpha,$$

where c_i, c_α and k_i are constants, $\hat{S}_{\pm\alpha}^{(k)}$, $\hat{S}_i^{(k)}$ are the components of the generalized spin operator living at the site k, and $\hat{a}_\alpha^\pm$ are the 'creation and annihilation operators' satisfying the same commutation relations as in the case of the quantum J–C model. The substitution of variables $\hat{a}_\alpha^\pm \equiv (-\alpha(K))^{1/2}\hat{b}_\alpha^\pm$ transforms them to the canonical form.

The second-order integrals $\hat{C}_0^2$ and $\hat{C}_{-k}^2$, defined for the above quantum Lax operator, acquire the following explicit form:

$$\hat{C}_0^2 = \sum_{k=1}^{N}\sum_{i=1}^{\operatorname{rank} g} k_i \hat{S}_i^{(k)} + \sum_{\alpha\in(\Delta/\Delta_0)_+} \frac{1}{2}\left(\hat{a}_\alpha^+\hat{a}_\alpha^- + \hat{a}_\alpha^-\hat{a}_\alpha^+\right) + \frac{1}{2}\sum_{i=1}^{\operatorname{rank} g} c_i^2 + \sum_{\alpha\in(\Delta_0)_+} c_\alpha c_{-\alpha}, \tag{21}$$

$$\begin{aligned}\hat{C}_{-k}^2 = {} & \sum_{i=1}^{\operatorname{rank} g} c_i \hat{S}_i^{(k)} + \sum_{\alpha\in\Delta_0} c_\alpha \hat{S}_\alpha^{(k)} + \sum_{\alpha\in(\Delta/\Delta_0)_+} \left(\hat{a}_\alpha^+\hat{S}_{-\alpha}^{(k)} + \hat{a}_\alpha^-\hat{S}_\alpha^{(k)}\right) \\ & + \nu_k \sum_{i=1}^{\operatorname{rank} g} k_i \hat{S}_i^{(k)} + \sum_{l=1, l\neq k}^{N} \frac{(\hat{S}^{(l)}, \hat{S}^{(k)})}{(\nu_l - \nu_k)}.\end{aligned} \tag{22}$$

The Hamiltonian of the generalized Dicke model is a linear combination of all these second-order integrals:

$$\hat{H} = \hat{C}_0^2 + g\sum_{l=1}^{N} \hat{C}_{-l}^2. \tag{23}$$

In terms of the canonical creation–annihilation operators $\hat{b}_\alpha^\pm \equiv (-\alpha(K))^{-1/2}\hat{a}_\alpha^\pm$ it acquires (up to a constant) the following explicit form:

$$\begin{aligned}\hat{H} = {} & \sum_{\alpha\in(\Delta/\Delta_0)_+} \left(\alpha(-K)\hat{b}_\alpha^+\hat{b}_\alpha^- + g\sqrt{\alpha(-K)}\sum_{l=1}^{N}\left(\hat{b}_\alpha^+\hat{S}_{-\alpha}^{(l)} + \hat{b}_\alpha^-\hat{S}_\alpha^{(l)}\right)\right) \\ & + \sum_{l=1}^{N}\sum_{i=1}^{\operatorname{rank} g} ((g\nu_l + 1)k_i + gc_i)\hat{S}_i^{(l)} + g\sum_{l=1}^{N}\sum_{\alpha\in\Delta_0} c_\alpha \hat{S}_\alpha^{(l)}.\end{aligned} \tag{24}$$

Remark 5. The generalized Jaynes–Cummings model is recovered as a partial case of the generalized Dicke model corresponding to the case $N = 1$, $\nu_1 = 0$.

Remark 6. In what follows we will put in the Hamiltonians and integrals of the generalized Jaynes–Cummings and Dicke models $c_\alpha = 0, \forall\alpha \in \Delta_0$. This requirement is necessary in order to diagonalize these Hamiltonians using the standard algebraic Bethe ansatz technique.

Let us consider the following example which will be the basis in the rest of the paper:

Example 4. Let us consider the case $\mathfrak{g} = gl(n)$ and $gl(n)_0 = gl(1) + \cdots + gl(1)$ i.e. element of the Cartan subalgebra $K = \sum_{i=1}^{n} k_i X_{ii}$ is generic (see example 2). The corresponding 'generic' Dicke Hamiltonian reads as follows:

$$\hat{H} = \sum_{i,j=1,i<j}^{n} (k_j - k_i)\hat{b}_{ij}^{+}\hat{b}_{ij}^{-} \pm g \sum_{l=1}^{N} \sum_{i,j=1,i<j}^{n} \sqrt{(k_j - k_i)}\big(\hat{b}_{ij}^{+}\hat{S}_{ji}^{(l)} + \hat{b}_{ij}^{-}\hat{S}_{ij}^{(l)}\big) + \sum_{l=1}^{N}\sum_{i=1}^{n}((g\nu_l + 1)k_i + gc_i)\hat{S}_{ii}^{(l)}, \tag{25}$$

where the generalized spin operators have Lie brackets isomorphic to $gl(n)^{\oplus N}$, $b_{ij}^{\pm}$ together with the unit operator form the Heisenberg algebra of dimension $n^2 - n + 1$:

$$\big[\hat{S}_{ij}^{(m)}, \hat{S}_{kl}^{(n)}\big] = \delta^{mn}\big(\delta_{kj}\hat{S}_{il}^{(m)} - \delta_{il}\hat{S}_{kj}^{(m)}\big), \tag{26}$$

$$\big[\hat{b}_{ij}^{+}, \hat{b}_{kl}^{-}\big] = \delta_{ik}\delta_{jl}\mathbf{1}, \tag{27}$$

$$\big[\hat{b}_{ij}^{+}, \hat{b}_{kl}^{+}\big] = \big[\hat{b}_{ij}^{-}, \hat{b}_{kl}^{-}\big] = \big[\hat{b}_{ij}^{+}, \hat{S}_{kl}^{(m)}\big] = \big[\hat{b}_{ij}^{-}, \hat{S}_{kl}^{(m)}\big] = 0. \tag{28}$$

4. Diagonalization

In this section, we will consider diagonalization of the Dicke model (the case of the Lax matrix with $N + 1$ poles) via an algebraic Bethe ansatz technique in the physically most interesting case $\mathfrak{g} = gl(n)$. We will not consider the Jaynes–Cummings model separately, viewing it as a partial case of the Dicke model.

The main tool of an algebraic Bethe ansatz approach is the algebra of the Lax operators, which has the same form for all models possessing the rational r-matrix of Yang (the Gaudin model, Gaudin model in the external magnetic field and Dicke models, etc). That is why we will proceed first purely algebraically and will fix the concrete form of the Lax operator corresponding to the Dicke model only at the end of the process of the diagonalization of the generating function of quantum integrals.

4.1. General case

Let us consider a case of $\mathfrak{g} = gl(n)$. In this case, we have the natural basic elements X_{ij}, $i, j \in 1, 2, \ldots, n$, $(X_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ with the standard commutation relations

$$[X_{ij}, X_{kl}] = \delta_{kj}X_{il} - \delta_{il}X_{kj}.$$

The quantum Lax matrix has the following form: $\hat{L}(u) = \sum_{i,j=1}^{n} \hat{L}_{ij}(u)X_{ij}$. The corresponding Lie bracket (7) among the components of the Lax operator is as follows:

$$[\hat{L}_{ij}(u), \hat{L}_{kl}(v)] = \frac{1}{(u-v)}(\delta_{kj}(\hat{L}_{il}(u) - \hat{L}_{il}(v)) - \delta_{il}(\hat{L}_{kj}(u) - \hat{L}_{kj}(v))). \tag{29}$$

In particular, the following commutation relations are important for the needs of the nested Bethe ansatz:

$$[\hat{L}_{11}(u), \hat{L}_{1l}(v)] = \frac{1}{(u-v)}(\hat{L}_{1l}(u) - \hat{L}_{1l}(v)), \qquad l > 1, \tag{30a}$$

$$[\hat{L}_{ij}(u), \hat{L}_{1l}(v)] = -\frac{1}{(u-v)}\delta_{il}(\hat{L}_{1j}(u) - \hat{L}_{1j}(v)), \qquad i, j, l > 1. \tag{30b}$$

$$[\hat{L}_{1j}(u), \hat{L}_{1l}(v)] = 0, \qquad j, l > 1. \tag{30c}$$

The generating operator of commutative second-order quantum integrals is

$$\hat{\tau}_n(u) \equiv \hat{C}^2(u) = \frac{1}{2}\sum_{i,j=1}^{n} \hat{L}_{ij}(u)\hat{L}_{ji}(u).$$

By direct calculation we obtain the following commutation relation of the generating function and components of the Lax operator:

$$[\hat{\tau}_n(u), \hat{L}_{kl}(v)] = \frac{1}{(u-v)}\sum_{m=1}^{n}(\hat{L}_{km}(v)\hat{L}_{ml}(u) - \hat{L}_{km}(u)\hat{L}_{ml}(v)). \tag{31}$$

This relation is essentially used while diagonalizing $\hat{\tau}(u)$.

Now let us diagonalize $\hat{\tau}_n(u)$ in a special representation space V with the help of the nested algebraic Bethe ansatz using the above-obtained formulae. Let V be the space of an irreducible representation of the algebra of Lax operators. Let us assume that there exist a vacuum vector $\mathbf{\Omega} \in V$ such that

$$\hat{L}_{ii}(u)\mathbf{\Omega} = \Lambda_{ii}(u)\mathbf{\Omega}, \qquad \hat{L}_{kl}(u)\mathbf{\Omega} = 0, \qquad \text{where} \quad i, k, l \in 1, 2, \ldots, n, \quad k > l, \tag{32}$$

and the whole space V be generated by the action of $\hat{L}_{kl}(u)$, $k < l$ on the vector $\mathbf{\Omega}$.

The following theorem gives us the spectrum of all integrable quantum systems possessing the rational $gl(n)$-valued r-matrix and the highest weight representation (32).

Theorem 4.1. *The spectrum of the generating function $\hat{\tau}_n(u)$ in the representation space V has the following explicit form:*

$$\begin{aligned}\Lambda_n(u) &= \frac{1}{2}\sum_{i=1}^{n}\Lambda_{ii}^2(u) - \frac{1}{2}\sum_{i=1}^{n}(n-2\mathrm{i}+1)\partial_u\Lambda_{ii}(u) - \Lambda_{11}(u)\sum_{i=1}^{M_1}\frac{1}{\left(u-v_i^{(1)}\right)} \\ &\quad + \sum_{k=2}^{n-1}\Lambda_{kk}(u)\left(\sum_{i=1}^{M_{k-1}}\frac{1}{\left(u-v_i^{(k-1)}\right)} - \sum_{i=1}^{M_k}\frac{1}{\left(u-v_i^{(k)}\right)}\right) + \Lambda_{nn}(u)\sum_{i=1}^{M_{n-1}}\frac{1}{\left(u-v_i^{(n-1)}\right)} \\ &\quad + \sum_{k=1}^{n-1}\sum_{i,j=1,i\neq j}^{M_k}\frac{1}{\left(u-v_i^{(k)}\right)\left(u-v_j^{(k)}\right)} + \sum_{k=1}^{n-1}\sum_{i=1}^{M_k}\sum_{j=1}^{M_{k+1}}\frac{1}{\left(u-v_i^{(k)}\right)\left(u-v_j^{(k+1)}\right)},\end{aligned} \tag{33}$$

where 'rapidities' $v_i^{(k)}$ satisfy the following Bethe-type equations:

$$\Lambda_{11}\left(v_i^{(1)}\right) - \Lambda_{22}\left(v_i^{(1)}\right) = 2\sum_{j=1;j\neq i}^{M_1}\frac{1}{\left(v_i^{(1)}-v_j^{(1)}\right)} - \sum_{j=1}^{M_2}\frac{1}{\left(v_i^{(1)}-v_j^{(2)}\right)}, \tag{34a}$$

$$\begin{aligned}\Lambda_{k+1k+1}\left(v_i^{(k+1)}\right) - \Lambda_{k+2k+2}\left(v_i^{(k+1)}\right) &= 2\sum_{j=1;j\neq i}^{M_{k+1}}\frac{1}{\left(v_i^{(k+1)}-v_j^{(k+1)}\right)} - \sum_{j=1}^{M_k}\frac{1}{\left(v_i^{(k+1)}-v_j^{(k)}\right)} \\ &\quad - \sum_{j=1}^{M_{k+2}}\frac{1}{\left(v_i^{(k+1)}-v_j^{(k+2)}\right)}, \qquad k \in 1, 2, \ldots, n-3,\end{aligned} \tag{34b}$$

$$\Lambda_{n-1n-1}\left(v_i^{(n-1)}\right) - \Lambda_{nn}\left(v_i^{(n-1)}\right) = 2\sum_{j=1;j\neq i}^{M_{n-1}}\frac{1}{\left(v_i^{(n-1)}-v_j^{(n-1)}\right)} - \sum_{j=1}^{M_{n-2}}\frac{1}{\left(v_i^{(n-1)}-v_j^{(n-2)}\right)}. \tag{34c}$$

Equations (34) can be viewed as a condition of the absence of poles of the function $\Lambda_n(u)$ at the points $v_i^{(k+1)}$:

$$\mathrm{res}_{u=v_i^{(k+1)}} \Lambda_n(u) = 0, \qquad k \in 0, 1, \ldots, n-2; \qquad i \in 1, 2, \ldots, M_{k+1}.$$

Sketch of the proof. The diagonalization of $\hat{\tau}_n(u)$ is performed on the special 'nested' Bethe vectors constructed with the help of the 'nested' Bethe ansatz. Its main idea consists in the recursion procedure based on the chain of the embeddings $gl(n) \supset gl(n-1) \supset \cdots \supset gl(2)$. We will not give a detailed description of all the recursion procedure due to its lengthy character. We will briefly describe only its first step.

In more detail, let us consider a subspace $V_0 \subset V$ consisting of the vectors $\mathbf{v}$ such that

$$\hat{L}_{11}(u)\mathbf{v} = \Lambda_{11}(u)\mathbf{v}, \qquad \hat{L}_{k1}(u)\mathbf{v} = 0, \qquad k > 1. \tag{35}$$

Using commutation relations in the algebra of the Lax operators it is easy to see that this subspace is invariant with respect to the action of the subalgebra of Lax operators taking values in the subalgebra $gl(n-1)$.

Let us consider the generating function $\hat{\tau}_n(u)$ and take into account that

$$\begin{aligned}
\hat{\tau}_n(u) = \frac{1}{2}\sum_{i,j=1}^{n} \hat{L}_{ij}(u)\hat{L}_{ji}(u) &= \frac{1}{2}\hat{L}_{11}^2(u) + \frac{1}{2}\sum_{j=2}^{n}(\hat{L}_{1j}(u)\hat{L}_{j1}(u) \\
&\quad + \hat{L}_{j1}(u)\hat{L}_{1j}(u)) + \frac{1}{2}\sum_{i,j=2}^{n}\hat{L}_{ij}(u)\hat{L}_{ji}(u) \\
&= \frac{1}{2}\hat{L}_{11}^2(u) + \sum_{j=2}^{n}\hat{L}_{1j}(u)\hat{L}_{j1}(u) + \frac{1}{2}\partial_u\left(\sum_{i=1}^{n}\hat{L}_{ii}(u) - n\hat{L}_{11}(u)\right) + \hat{\tau}_{n-1}(u),
\end{aligned} \tag{36}$$

where $\hat{\tau}_{n-1}(u) = \frac{1}{2}\sum_{i,j=2}^{n}\hat{L}_{ij}(u)\hat{L}_{ji}(u)$ is the generating function of commutative integrals of a $gl(n-1)$-valued subalgebra of Lax operators. Using formula (36) we obtain the following action of the 'full' generating function on the vector $\mathbf{v}$:

$$\hat{\tau}_n(u)\mathbf{v} = \frac{1}{2}\big(\Lambda_{11}^2(u) - n\partial_u\Lambda_{11}(u)\big)\mathbf{v} + \frac{1}{2}\partial_u\left(\sum_{i=1}^{n}\hat{L}_{ii}(u)\right)\mathbf{v} + \hat{\tau}_{n-1}(u)\mathbf{v}.$$

Due to the fact that $\hat{C}_n^1(u) = \sum_{i=1}^{n}\hat{L}_{ii}(u)$ is a Casimir operator, which is constant in all irreducible representations, we have that $\partial_u\big(\sum_{i=1}^{n}\hat{L}_{ii}(u)\big)\mathbf{v} = \partial_u\big(\sum_{i=1}^{n}\Lambda_{ii}(u)\big)\mathbf{v}$. Hence, in order to diagonalize the generating function $\hat{\tau}_n(u)$ of the $gl(n)$-valued Lax operators one has to do this also for the generating functions $\hat{\tau}_{n-1}(u)$ of the $gl(n-1)$-valued Lax operators. At this point one comes to the next idea used in the nested Bethe ansatz. Namely, in order to have a correct Bethe ansatz one has to diagonalize on the next step of the recursive procedure not $\hat{\tau}_{n-1}(u)$ acting in the space V_0, but some other generating function $\hat{\tau}_{n-1}^{(1)}(u)$ acting in the space $V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M}$.

In more detail, let us consider special vectors in $V \otimes (\mathbb{C}^n)^{\otimes M}$ of the form

$$\mathbf{v}^{(1)} = \sum_{i_1,\ldots,i_M=2}^{n} v_{i_1 i_2\ldots i_M} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_M}$$

such that each $v_{i_1 i_2\ldots i_M}$ belongs to the space V_0, and e_i are basis vectors in the space $\mathbb{C}^n$ (vector columns with unit on the place i and zeros everywhere else). These special vectors constitute the subspace $V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M}$. Let us define the following operator-valued matrix (part of the Lax matrix):

$$\hat{B}(u) = \sum_{j=2}^{n} \hat{L}_{1j}(u) X_{1j}.$$

We will need to 'lift' this matrix with the non-commuting entries to the tensor products of the M copies of $gl(n)$, and use the following 'tensorial' notations: $\hat{B}_k(v_k) \equiv 1_n \otimes \cdots \otimes \hat{B}(v_k) \otimes \cdots \otimes 1_n$ where $\hat{B}(v_k)$ stands in the kth component of the tensor product, and 1_n is the unit matrix in the space $\mathbb{C}^n$. We will also lift the action of $\hat{\tau}_n(u)$ to the tensor product $V \otimes (\mathbb{C}^n)^{\otimes M}$ in the following trivial way:

$$\hat{T}_M(u) \equiv \hat{\tau}_n(u) 1_n \otimes \cdots \otimes 1_n \otimes \cdots \otimes 1_n. \tag{37}$$

It is evident that operators $\hat{T}_M(u)$ on $V \otimes (\mathbb{C}^n)^{\otimes M}$ and $\hat{\tau}_n(u)$ on V act in the same way and have the same spectrum. This permits to consider the problem of diagonalization of $\hat{T}_M(u)$ on $V \otimes (\mathbb{C}^n)^{\otimes M}$ instead of the problem of diagonalization of $\hat{\tau}_n(u)$ on V.

Let us now consider the Bethe-type vectors of the following form:

$$\mathbf{v}\big(v_1^{(1)}, \ldots, v_M^{(1)}\big) = \hat{B}_1\big(v_1^{(1)}\big) \cdots \hat{B}_M\big(v_M^{(1)}\big) \mathbf{v}^{(1)}. \tag{38}$$

Let us diagonalize the operator $\hat{T}_M(u)$ on $V \otimes (\mathbb{C}^n)^{\otimes M}$ using the Bethe vectors (38). By long and tedious calculations, using the commutation relations (30) and (31) we obtain the following action formula:

$$\begin{aligned}
\hat{T}_M(u)\mathbf{v}\big(v_1^{(1)}, \ldots, v_M^{(1)}\big) = {} & \left(\frac{1}{2}\Lambda_{11}^2(u) + \frac{1}{2}\partial_u\left(\sum_{i=1}^{n}\Lambda_{ii}(u) - n\Lambda_{11}(u)\right) - \sum_{i=1}^{M}\frac{\Lambda_{11}(u)}{\big(u - v_i^{(1)}\big)}\right. \\
& \left. - \sum_{i=1}^{M}\frac{(n-1)}{2\big(u - v_i^{(1)}\big)^2} + \sum_{i,j=1;i<j}^{M}\frac{1}{\big(u - v_i^{(1)}\big)\big(u - v_j^{(1)}\big)}\right)\mathbf{v}\big(v_1^{(1)}, \ldots, v_M^{(1)}\big) \\
& + \hat{B}_1\big(v_1^{(1)}\big) \cdots \hat{B}_M\big(v_M^{(1)}\big)\hat{\tau}_{n-1}^{(1)}(u)\mathbf{v}^{(1)} + \sum_{i=1}^{M}\left(\Lambda_{11}\big(v_i^{(1)}\big) - \sum_{j=1;j\neq i}^{M}\frac{1}{\big(v_i^{(1)} - v_j^{(1)}\big)}\right. \\
& \left. - \operatorname{res}_{u=v_i}\hat{\tau}_{n-1}^{(1)}(u)\right)\frac{\hat{B}_1\big(v_1^{(1)}\big) \cdots \hat{B}_i(u) \cdots \hat{B}_M\big(v_M^{(1)}\big)}{\big(u - v_i^{(1)}\big)}\mathbf{v}^{(1)},
\end{aligned} \tag{39}$$

where $\hat{\tau}_{n-1}^{(1)}(u) = \frac{1}{2}\operatorname{tr}\big(\hat{L}_{n-1}^{(1)}(u)\big)^2$, and $\hat{L}_{n-1}^{(1)}(u)$ is defined as follows:

$$\hat{L}_{n-1}^{(1)}(u) \equiv \sum_{i,j=2}^{n}\left(\hat{L}_{ij}(u) + \sum_{k=1}^{M}\frac{X_{ji}^{(k)}}{\big(u - v_k^{(1)}\big)}\right) X_{ij}.$$

Here $X_{ij}^{(k)}$ acts as X_{ij} in the kth multiplier of $C^{\otimes M}$ and as the unit operator in other multipliers. It is easy to show that both $\hat{L}_{n-1}^{(0)}(u) \equiv \sum_{i,j=2}^{n}\hat{L}_{ij}(u)X_{ij}$ and $\hat{L}_{n-1}^{(1)}(u)$ satisfy the linear r-matrix brackets with the same $gl(n-1)$-valued rational r-matrix.

From the action formula (39) it is evident that the vector $\mathbf{v}\big(v_1^{(1)}, \ldots, v_M^{(1)}\big)$ is an eigenvector for $\hat{T}_M(u)$ with the eigenvalue $\Lambda_n(u)$, where

$$\begin{aligned}
\Lambda_n(u) = {} & \left(\frac{1}{2}\left(\Lambda_{11}^2(u) + \partial_u\left(\sum_{i=1}^{n}\Lambda_{ii}(u) - n\Lambda_{11}(u)\right)\right) - \sum_{i=1}^{M}\frac{\Lambda_{11}(u)}{\big(u - v_i^{(1)}\big)}\right. \\
& \left. - \frac{(n-1)}{2}\sum_{i=1}^{M}\frac{1}{\big(u - v_i^{(1)}\big)^2} + \sum_{i,j=1;i<j}^{M}\frac{1}{\big(u - v_i^{(1)}\big)\big(u - v_j^{(1)}\big)} + \Lambda_{n-1}^{(1)}(u)\right),
\end{aligned}$$

if $\mathbf{v}^{(1)}$ is an eigenvector for $\hat{\tau}^{(1)}_{n-1}(u)$ with the eigenvalue $\Lambda^{(1)}_{n-1}(u)$:

$$\hat{\tau}^{(1)}_{n-1}(u)\mathbf{v}^{(1)} = \Lambda^{(1)}_{n-1}(u)\mathbf{v}^{(1)},$$

and the following Bethe equations are satisfied:

$$\Lambda_{11}\big(v_i^{(1)}\big) - \sum_{j=1;\, j\neq i}^{M} \frac{1}{\big(v_i^{(1)} - v_j^{(1)}\big)} - \operatorname{res}_{u=v_i^{(1)}} \Lambda^{(1)}_{n-1}(u) = 0. \tag{40}$$

This reduces the problem of finding the spectrum of the generating function $\hat{\tau}_n(u)$ of the quantum integrals for the $gl(n)$-valued Lax operators to the problem of finding the spectrum of $\hat{\tau}^{(1)}_{n-1}(u)$ of the generating function of the quantum integrals for the $gl(n-1)$-valued Lax operator possessing additional (in comparison to the initial Lax operator) first-order poles in the points $v_i^{(1)}, i \in 1, 2, \ldots, M$ that solve Bethe equations (40). Proceeding further we come to the $(n-2)$ step of the recursion process to the problem of diagonalization of the generating function of the $gl(2)$-valued Lax operator. It is diagonalized with the help of an ordinary algebraic Bethe ansatz, and all the answers for this are known [6]. Substituting them in the obtained result of our recursion procedure formulae one finally obtains the formulae (33) and (34).

This finishes our sketch of the proof. □

4.2. Case of the Dicke model

Let us consider a representation of the algebra of Lax operators in a Hilbert space V that corresponds to the algebra $gl(n)^{\oplus N} \oplus H_K$, where H_K is described in the previous sections Heisenberg algebra, depending on the element $K = \sum_{i=1}^n k_i X_{ii}$. In more detail,

$$\big[\hat{S}^{(m)}_{ij}, \hat{S}^{(n)}_{kl}\big] = \delta^{mn}\big(\delta_{kj}\hat{S}^{(m)}_{il} - \delta_{il}\hat{S}^{(m)}_{kj}\big), \tag{41}$$

$$\big[\hat{a}^+_{ij}, \hat{a}^-_{kl}\big] = (k_j - k_i)\delta_{ik}\delta_{jl}\mathbf{1}, \tag{42}$$

$$\big[\hat{a}^+_{ij}, \hat{a}^+_{kl}\big] = \big[\hat{a}^-_{ij}, \hat{a}^-_{kl}\big] = \big[\hat{a}^+_{ij}, \hat{S}^{(m)}_{kl}\big] = \big[\hat{a}^-_{ij}, \hat{S}^{(m)}_{kl}\big] = 0, \tag{43}$$

and $\hat{a}^+_{ij} = \sqrt{(k_j - k_i)}\hat{b}^+_{ij}, \hat{a}^-_{ij} = \sqrt{(k_j - k_i)}\hat{b}^-_{ij}$ where $\hat{b}^\pm_{ij}$ are the canonical creation–annihilation operators.

Remark 7. Note that in the case of the degeneration of the element K, when $k_i = k_j$ for some indices i and j we have that $\hat{a}^\pm_{ij} = 0$, i.e. dimension of the corresponding Heisenberg algebra decreases.

The Lax operator of the Dicke model in the case $c_{\alpha_{ij}} = 0, \alpha_{ij} \in \Delta_0$ has the form

$$\hat{L}(u) = u\hat{L}^{(-2)} + \hat{L}^{(-1)} + \sum_{m=1}^{N} \frac{\hat{L}^{(m)}}{u - \nu_m}, \qquad \text{where} \quad \hat{L}^{(-2)} = \sum_{i=1}^{n} k_i X_{ii},$$

$$\hat{L}^{(-1)} = \sum_{i=1}^{n} c_i X_{ii} + \sum_{i,j=1, i<j}^{n} \sqrt{(k_j - k_i)}\, \hat{b}^+_{ij} X_{ji} + \sum_{i,j=1, i<j}^{n} \sqrt{(k_j - k_i)}\, \hat{b}^-_{ij} X_{ij},$$

$$\hat{L}^{(m)} = \sum_{i,j=1}^{n} \hat{S}^{(m)}_{ij} X_{ji}.$$

A space of the irreducible representations of the algebra of Lax operators will have the form $V = \left(\otimes_{i=1}^{N} V^{\lambda^{(i)}}\right) \otimes V^{H_K}$, where $V^{\lambda^{(i)}}$ is a space of an irreducible representation of $gl(n)^{(m)}$ label by the highest weight vectors $\lambda^{(m)} = \left(\lambda_1^{(m)}, \ldots, \lambda_n^{(m)}\right)$:

$$\hat{S}_{ii}^{(m)} \mathbf{\Omega}_m = \lambda_i^{(m)} \mathbf{\Omega}_m, \qquad \hat{S}_{kl}^{(m)} \mathbf{\Omega}_m = 0, \qquad \text{where} \quad i, k, l \in 1, 2, \ldots, n, \\ k < l; m \in 1, 2, \ldots, N, \tag{44}$$

V^{H_K} is an irreducible representation of the algebra H_K with the vacuum vector $\mathbf{\Omega}_0$:

$$\hat{b}_{kl}^{+} \mathbf{\Omega}_0 = 0, \qquad \text{where} \quad k, l \in 1, 2, \ldots, n, \quad k < l. \tag{45}$$

(Note that for such a definition of vacuum vector operators $\hat{b}_{kl}^{+}$ play the role of 'annihilation' operators and operators $\hat{b}_{kl}^{-}$ play the role of 'creation' operators.)

It is easy to see that the 'vacuum' vector satisfying (32) for the corresponding representation of the Lax algebra exists and has the form

$$\mathbf{\Omega} = \mathbf{\Omega}_0 \otimes \mathbf{\Omega}_1 \otimes \cdots \otimes \mathbf{\Omega}_N.$$

Hence, we can apply in the case of the Lax operators of the Dicke model the whole technique of the nested algebraic Bethe ansatz, which was described above. We have that in our case

$$\Lambda_{ii}(u) = uk_i + c_i + \sum_{m=1}^{N} \frac{\lambda_i^{(m)}}{u - \nu_m}, \tag{46}$$

where $\hat{L}_{ii}(u)\mathbf{\Omega} = \Lambda_{ii}(u)\mathbf{\Omega}$. Substituting this expression in theorem 4.1 we obtain the explicit form of the spectrum of the generating function $\hat{\tau}_n(u)$ in the case of the generalized Dicke model.

We will explicitly calculate the spectrum of the integrals $\hat{C}_0^2$ and $\hat{C}_{-m}^2$. The following corollary of theorem 4.1 holds true.

Corollary 4.1. *The spectrum of the integrals $\hat{C}_0^2$ and $\hat{C}_{-m}^2$ on the Bethe-type vectors has the following explicit form:*

$$c_0^2(M_1, \ldots, M_{n-1}) = \sum_{i=1}^{n} \left(\frac{c_i^2}{2} + \sum_{m=1}^{N} k_i \lambda_i^{(m)} - \frac{1}{2}(n - 2\mathrm{i} + 1) k_i \right) \\ - k_1 M_1 + \sum_{i=2}^{n-1} k_i (M_{i-1} - M_i) + k_n M_{n-1}, \tag{47}$$

$$c_{-m}^2\left(\left\{v_i^{(1)}\right\}, \ldots, \left\{v_i^{(n-1)}\right\}\right) = \left(\sum_{i=1}^{n} (\nu_m k_i + c_i) \lambda_i^{(m)} + \sum_{l=1, l\neq m}^{N} \frac{\lambda_i^{(m)} \lambda_i^{(l)}}{\nu_m - \nu_l} \right) - \lambda_1^{(m)} \sum_{i=1}^{M_1} \frac{1}{\nu_m - v_i^{(1)}} \\ + \sum_{k=2}^{n-1} \lambda_k^{(m)} \left(\sum_{i=1}^{M_{k-1}} \frac{1}{\nu_m - v_i^{(k-1)}} - \sum_{i=1}^{M_k} \frac{1}{\nu_m - v_i^{(k)}} \right) + \lambda_n^{(m)} \sum_{i=1}^{M_{n-1}} \frac{1}{\nu_m - v_i^{(n-1)}}, \tag{48}$$

where rapidities $v_i^{(k)}$, $i \in 1, 2, \ldots, M_k$ *satisfy the following Bethe-type equations:*

$$v_i^{(1)}(k_1 - k_2) + (c_1 - c_2) + \sum_{k=1}^{N} \frac{\left(\lambda_1^{(k)} - \lambda_2^{(k)}\right)}{v_i^{(1)} - \nu_k} = \sum_{j=1;\, j\neq i}^{M_1} \frac{2}{\left(v_i^{(1)} - v_j^{(1)}\right)} - \sum_{j=1}^{M_2} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)},$$

$$v_i^{(m+1)}(k_{m+1} - k_{m+2}) + (c_{m+1} - c_{m+2}) + \sum_{k=1}^{N} \frac{\left(\lambda_{m+1}^{(k)} - \lambda_{m+2}^{(k)}\right)}{v_i^{(m+1)} - \nu_k}$$
$$= \sum_{j=1;\, j\neq i}^{M_{m+1}} \frac{2}{\left(v_i^{(m+1)} - v_j^{(m+1)}\right)} - \sum_{j=1}^{M_m} \frac{1}{\left(v_i^{(m+1)} - v_j^{(m)}\right)} - \sum_{j=1}^{M_{m+2}} \frac{1}{\left(v_i^{(m+1)} - v_j^{(m+2)}\right)},$$
$$m \in 1, 2, \ldots, n-3,$$

$$v_i^{(n-1)}(k_{n-1} - k_n) + (c_{n-1} - c_n) + \sum_{k=1}^{N} \frac{\left(\lambda_{n-1}^{(k)} - \lambda_n^{(k)}\right)}{v_i^{(n-1)} - \nu_k}$$
$$= \sum_{j=1;\, j\neq i}^{M_{n-1}} \frac{2}{\left(v_i^{(n-1)} - v_j^{(n-1)}\right)} - \sum_{j=1}^{M_{n-2}} \frac{1}{\left(v_i^{(n-1)} - v_j^{(n-2)}\right)}.$$

Remark 8. The spectrum of the generalized Dicke Hamiltonian (2) on the Bethe-type vectors is easily found from the above corollary and has (up to a non-sufficient constant $c = \sum_{i=1}^{n} \left(\frac{c_i^2}{2} - \sum_{m=1}^{N} (n - 2\mathrm{i} + 1)k_i\right)$) the following form:

$$h\left(\left\{v_i^{(1)}\right\}, \ldots, \left\{v_i^{(n-1)}\right\}\right) = c_0^2(M_1, \ldots, M_{n-1}) + g \sum_{m=1}^{N} c_{-m}^2\left(\{v_i^{(1)}\}, \ldots, \left\{v_i^{(n-1)}\right\}\right).$$

Remark 9. All the 'reduced' Dicke models described in this paper may be considered as limiting cases of the 'full' Dicke model corresponding to the generic element K and the maximal number of bosonic fields. The difference will be in the number of bosonic fields entered into the definition of the Lax operator, form of the Bethe vectors and in the form of corresponding Bethe equations. Indeed, from the explicit form of the Bethe equations it is easy to see that in the case of the degeneration, for example, when $k_{m+1} = k_{m+2}$ the corresponding subset of the Bethe equations becomes simpler and coincides with the Bethe equation of the Gaudin system in an external magnetic field and if, moreover $c_{m+1} = c_{m+2}$, with the Bethe equation of an ordinary Gaudin system.

Example 5. Let us consider the particular example of the spectrum and Bethe equations of the three-level ($n = 3$) N-atom Dicke model corresponding to the case of $gl(3)$. The spectrum of the second-order integrals is the following:

$$c_0^2(M_1, M_2) = \sum_{i=1}^{3} \left(\frac{c_i^2}{2} + \sum_{m=1}^{N} k_i \lambda_i^{(m)} - (2 - \mathrm{i})k_i \right) - k_1 M_1 + k_2(M_1 - M_2) + k_3 M_2,$$

$$c_{-m}^2\left(\left\{v_i^{(1)}\right\}, \left\{v_i^{(2)}\right\}\right) = \left(\sum_{i=1}^{3} (\nu_m k_i + c_i)\lambda_i^{(m)} + \sum_{l=1, l\neq m}^{N} \frac{\lambda_i^{(m)}\lambda_i^{(l)}}{\nu_m - \nu_l} \right) - \lambda_1^{(m)} \sum_{i=1}^{M_1} \frac{1}{\nu_m - v_i^{(1)}}$$
$$+ \lambda_2^{(m)} \left(\sum_{i=1}^{M_1} \frac{1}{\nu_m - v_i^{(1)}} - \sum_{i=1}^{M_2} \frac{1}{\nu_m - v_i^{(2)}} \right) + \lambda_3^{(m)} \sum_{i=1}^{M_2} \frac{1}{\nu_m - v_i^{(2)}},$$

where rapidities $v_i^{(k)}$, $i \in 1, 2, \ldots, M_k$, $k = 1, 2$ satisfy the following Bethe equations:

$$v_i^{(1)}(k_1 - k_2) + (c_1 - c_2) + \sum_{k=1}^{N} \frac{\left(\lambda_1^{(k)} - \lambda_2^{(k)}\right)}{v_i^{(1)} - \nu_k} = \sum_{j=1; j\neq i}^{M_1} \frac{2}{\left(v_i^{(1)} - v_j^{(1)}\right)} - \sum_{j=1}^{M_2} \frac{1}{\left(v_i^{(1)} - v_j^{(2)}\right)},$$

$$v_i^{(2)}(k_2 - k_3) + (c_2 - c_3) + \sum_{k=1}^{N} \frac{\left(\lambda_2^{(k)} - \lambda_3^{(k)}\right)}{v_i^{(2)} - \nu_k} = \sum_{j=1; j\neq i}^{M_2} \frac{2}{\left(v_i^{(2)} - v_j^{(2)}\right)} - \sum_{j=1}^{M_1} \frac{1}{\left(v_i^{(2)} - v_j^{(1)}\right)}.$$

5. Conclusion and discussion

Using the technique of classical rational r-matrices and rational Lax operators we have constructed 'n-level many-mode' integrable generalizations of Jaynes–Cummings and Dicke Hamiltonians labeled by a pair of semisimple (reductive) Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.

All the constructed n-level many-mode Jaynes–Cummings or Dicke-type Hamiltonians may be obtained as a reduction of the n-level Jaynes–Cummings or Dicke-type Hamiltonians with a maximal number of bosons. In particular, in the case of $\mathfrak{g} = gl(n)$ the maximally reduced generalized n-level Jaynes–Cummings model is equivalent to the so-called n-level, $n-1$-mode Jaynes–Cummings model in the Λ-configuration [11–13].

In the case of the Lie algebra $\mathfrak{g} = gl(n)$ the obtained Hamiltonians may be interpreted as the approximate Hamiltonians of the n-level atom interacting with an electromagnetic field or Hamiltonians of the set of N n-level atoms interacting with an electromagnetic field, respectively. In this case, we have calculated the spectra of all of the constructed Hamiltonians and the corresponding commuting integrals using the nested algebraic Bethe ansatz technique.

It would be very interesting to find other physical quantities of this model using the developed technique of the nested Bethe ansatz. In particular, it seems to be possible to construct correlation functions of the generalized Jaynes–Cummings and Dicke models using the r-matrix and Bethe ansatz technique. For this purpose it will be necessary to generalize the approach of [27] from an ordinary to the nested Bethe ansatz case. We plan to return to this problem in our subsequent publications.

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